6.3

1-6. Write out the form of the partial fraction decomposition of the function. Do not determine the numerical values of the coefficients.

1. (a) $\dfrac{2x}{(x+3)(3x+1)}$ (b) $\dfrac{1}{x^3+2x^2+x}$

(a) $\dfrac{2x}{(x+3)(3x+1)}=\dfrac{\text{A}}{x+3}+\dfrac{\text{B}}{3x+1}$

(b) $\dfrac{1}{x^3+2x^2+x}=\dfrac{1}{x(x+1)^2}=\dfrac{\text{A}}{x}+\dfrac{\text{B}}{(x+1)^2}+\dfrac{\text{C}}{x+1}$

3. (a) $\dfrac{1}{x^2+3x-4}$ (b) $\dfrac{x^2}{(x-1)(x^2+x+1)}$

(a) $\dfrac{1}{x^2+3x-4}=\dfrac{1}{(x+4)(x-1)}=\dfrac{\text{A}}{x+4}+\dfrac{\text{B}}{x-1}$

(b) $\dfrac{x^2}{(x-1)(x^2+x+1)}=\dfrac{\text{A}}{x-1}+\dfrac{\text{B}x+\text{C}}{x^2+x+1}$

5. (a) $\dfrac{x^4}{x^4-1}$ (b) $\dfrac{t^4+t^2+1}{(t^2+1)(t^2+4)^2}$

(a) $\dfrac{x^4}{x^4-1}=1+\dfrac{1}{x^4-1}=1+\dfrac{1}{(x-1)(x+1)(x^2+1)}=1+\dfrac{\text{A}}{x-1}+\dfrac{\text{B}}{x+1}+\dfrac{\text{C}x+\text{D}}{x^2+1}$

(b) $\dfrac{t^4+t^2+1}{(t^2+1)(t^2+4)^2}=\dfrac{\text{A}t+\text{B}}{t^2+1}+\dfrac{\text{C}t+\text{D}}{(t^2+4)^2}+\dfrac{\text{E}t+\text{F}}{t^2+4}$

7-34. Evaluate the integral.

7. $\displaystyle\int\frac{x}{x-6}\,dx$

$\displaystyle\Rightarrow\int\left(1+\frac{6}{x-6}\right)dx=x+6\ln|x-6|+\text{C}$

11. $\displaystyle\int_2^3\frac{1}{x^2-1}\,dx$

$\displaystyle\Rightarrow\frac{1}{2}\int_2^3\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx=\frac{1}{2}\left[\ln\left|\frac{x-1}{x+1}\right|\right]_2^3=\frac{1}{2}\ln\frac{3}{2}$

15. $\displaystyle\int_0^1\frac{2x+3}{(x+1)^2}\,dx$

$\displaystyle\Rightarrow\int_0^1\frac{2}{x+1}\,dx+\int_0^1\frac{1}{(x+1)^2}\,dx=\left[2\ln|x+1|-\frac{1}{x+1}\right]_0^1=2\ln2+\frac{1}{2}$

21. $\displaystyle\int\frac{5x^2+3x-2}{x^3+2x^2}\,dx$

$\displaystyle\Rightarrow\int\frac{2}{x}\,dx-\int\frac{1}{x^2}\,dx+\int\frac{3}{x+2}\,dx=2\ln|x|+\frac{1}{x}+3\ln|x+2|+\text{C}$

35-40. Make a substitution to express the integrand as a rational function and then evaluate the integral.

35. $\int_9^{16} \frac{\sqrt{x}}{x-4} dx$

令 $t=\sqrt{x} \Rightarrow dt = \frac{dx}{2\sqrt{x}}$

則 $\int_9^{16} \frac{\sqrt{x}}{x-4} dx = 2\int_3^4 \frac{t^2}{t^2-4} dt = 2\int_3^4 \left(1+\frac{4}{t^2-4}\right) dt = 2\int_3^4 \left(1+\frac{1}{t-2}-\frac{1}{t+2}\right) dt$

$$= 2\left(t + \ln|t-2 - \ln|t+2|\right)\Big|_3^4 = 2+\ln\frac{25}{9}$$

39. $\int \frac{e^{2x}}{e^{2x}+3e^x+2} dx$

令 $t=e^x \Rightarrow dt = e^x dx$

則 $\int \frac{e^{2x}}{e^{2x}+3e^x+2} dx = \int \frac{t}{t^2+3t+2} dt = \int \frac{2}{t+2} dt - \int \frac{1}{t+1} dt$

$$= 2\ln|t+2| - \ln|t+1| + C = \ln\frac{(t+2)^2}{|t+1|} + C = \ln\frac{(e^x+2)^2}{e^x+1} + C$$

41. $\int \ln(x^2-x+2)dx$. Use integration by parts, together with the techniques of this section, to evaluate the integral.

令 $u=\ln(x^2-x+2), v'=1 \Rightarrow u' = \frac{2x-1}{x^2-x+2}, v=x$

則 $\int \ln(x^2-x+2)dx = x\ln(x^2-x+2) - \int \frac{2x^2-x}{x^2-x+2} dx$

$$= x\ln(x^2-x+2) - \int \left(2+\frac{x-4}{x^2-x+2}\right) dx$$

$$= x\ln(x^2-x+2) - 2x - \frac{1}{2}\int \frac{2x-1}{x^2-x+2} dx + \int \left(\frac{\frac{7}{2}}{\left(x-\frac{1}{2}\right)^2+\frac{7}{4}}\right) dx$$

其中，$\int \frac{2x-1}{x^2-x+2} dx = \ln(x^2-x+2)$；$\int \left(\frac{\frac{7}{2}}{\left(x-\frac{1}{2}\right)^2+\frac{7}{4}}\right) dx = \sqrt{7}\tan^{-1}\left(\frac{2x-1}{\sqrt{7}}\right)$

所以 $\int \ln(x^2-x+2)dx = \left(x-\frac{1}{2}\right)\ln(x^2-x+2) - 2x + \sqrt{7}\tan^{-1}\left(\frac{2x-1}{\sqrt{7}}\right) + \mathrm{C}$

6.6

1. Explain why each of the following integrals is improper.

(a) $\int_1^\infty x^4 e^{-x^4} dx$ (b) $\int_0^{\pi/2} \sec x dx$ (c) $\int_0^2 \frac{x}{x^2-5x+6} dx$ (d) $\int_{-\infty}^0 \frac{1}{x^2+5} dx$

(a) $\int_1^\infty x^4 e^{-x^4} dx$ 積分區間為無窮大，為第一型瑕積分

(b) $\sec x$ 在 $x=\frac{\pi}{2}$ 時為無窮大，為第二型瑕積分

(c) $\frac{x}{x^2-5x+6}$ 在 $x=2$ 時為無窮大，為第二型瑕積分

(d) $\int_{-\infty}^0 \frac{1}{x^2+5} dx$ 積分區間為無窮大，為第一型瑕積分

5-32. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

5. $\int_1^\infty \frac{1}{(3x+1)^2} dx$

$$\Rightarrow \lim_{t\to\infty}\int_1^t \frac{1}{(3x+1)^2} dx = \lim_{t\to\infty}\left[-\frac{1}{3}(3x+1)^{-1}\right]_0^t = \frac{1}{12}$$ 積分收斂

13. $\int_{-\infty}^\infty xe^{-x^2} dx$

$$\int_{-\infty}^\infty xe^{-x^2} dx = \int_{-\infty}^0 xe^{-x^2} dx + \int_0^\infty xe^{-x^2} dx$$

$$\int xe^{-x^2} dx = -\frac{1}{2}e^{-x^2} + C \Rightarrow \int_{-\infty}^0 xe^{-x^2} dx + \int_0^\infty xe^{-x^2} dx = -\frac{1}{2}+\frac{1}{2} = 0$$ 積分收斂

17. $\int_1^\infty \frac{\ln x}{x} dx$

$$\Rightarrow \lim_{t\to\infty}\int_1^t \frac{\ln x}{x} dx = \lim_{t\to\infty}\left[\frac{(\ln x)^2}{2}\right]_0^t = \infty$$ 積分發散

21. $\int_{-\infty}^\infty \frac{x^2}{9+x^6} dx$

$$\Rightarrow \int_{-\infty}^\infty \frac{x^2}{9+x^6} dx = \int_{-\infty}^0 \frac{x^2}{9+x^6} dx + \int_0^\infty \frac{x^2}{9+x^6} dx = \lim_{s\to-\infty}\int_s^0 \frac{x^2}{9+x^6} dx + \lim_{t\to\infty}\int_0^t \frac{x^2}{9+x^6} dx$$

令 $u=\frac{1}{3}x^3 \Rightarrow du = x^2dx$ ，則 $\int \frac{x^2}{9+x^6} dx = \frac{1}{9}\int \frac{du}{1+u^2} = \frac{1}{9}\tan^{-1} u + C = \frac{1}{9}\tan^{-1}\frac{x^3}{3} + C$

所以 $\lim_{s\to-\infty}\int_s^0 \frac{x^2}{9+x^6} dx + \lim_{t\to\infty}\int_0^t \frac{x^2}{9+x^6} dx = \frac{1}{9}\left[\lim_{t\to-\infty}\tan^{-1}\frac{t^3}{3} - \lim_{s\to-\infty}\tan^{-1}\frac{s^3}{3}\right] = \frac{1}{9}\pi$ 積分收斂

25. $\int_{-2}^{14}\frac{1}{\sqrt[4]{x+2}}dx$

$\Rightarrow \lim_{t\to -2^+}\int_t^{14}(x+2)^{-1/4}dx=\frac{4}{3}(14+2)^{3/4}-\lim_{t\to -2^+}\frac{4}{3}(x+2)^{3/4}=\frac{32}{3}$ 積分收斂

29. $\int_{-1}^{1}\frac{e^x}{e^x-1}dx$

$\Rightarrow \int_{-1}^{1}\frac{e^x}{e^x-1}dx=\int_{-1}^{0}\frac{e^x}{e^x-1}dx+\int_{0}^{1}\frac{e^x}{e^x-1}dx=\lim_{s\to 0^-}\int_{-1}^{s}\frac{e^x}{e^x-1}dx+\lim_{t\to 0^+}\int_{t}^{1}\frac{e^x}{e^x-1}dx$

$\int\frac{e^x}{e^x-1}dx=\ln|e^x-1|+C$

$\Rightarrow \lim_{s\to 0^-}\int_{-1}^{s}\frac{e^x}{e^x-1}dx+\lim_{t\to 0^+}\int_{t}^{1}\frac{e^x}{e^x-1}dx=\lim_{s\to 0^-}\ln|e^s-1|-\ln|e^{-1}-1|+\ln|e^1-1|-\lim_{t\to 0^+}\ln|e^t-1|$
$=-\infty+\infty$(不定型) 積分發散

41-46. Use the Comparison Theorem to determine whether the integral is convergent or divergent.

43. $\int_{1}^{\infty}\frac{1}{x+e^{2x}}dx$

對於 $x\geq 1\Rightarrow \frac{1}{x+e^{2x}}\leq\frac{1}{e^{2x}}\Rightarrow \int_1^{\infty}\frac{1}{x+e^{2x}}dx\leq\int_1^{\infty}\frac{1}{e^{2x}}dx=\lim_{t\to\infty}\frac{-1}{2}e^{-2t}+\frac{1}{2}e^{-2}=\frac{1}{2}e^{-2}$

依比較定理，該積分收斂。

45. $\int_{0}^{\pi/2}\frac{1}{x\sin x}dx$

對於 $0\leq x\leq\frac{\pi}{2}\Rightarrow \sin x\leq 1\Rightarrow\frac{1}{x\sin x}\geq\frac{1}{x}\Rightarrow\int_0^{\pi/2}\frac{1}{x\sin x}dx\geq\int_0^{\pi/2}\frac{1}{x}dx=\ln\frac{\pi}{2}-\lim_{t\to 0^+}\ln t=\infty$

依比較定理，該積分發散

47. The integral $\int_0^{\infty}\frac{1}{\sqrt{x}(1+x)}dx$ is improper for two reasons: The interval $[0,\infty)$ is infinite and the integrand has an infinite discontinuity at 0. Evaluate it by expressing it as a sum of improper integrals of Type 2 and Type 1 as follows: $\int_0^{\infty}\frac{1}{\sqrt{x}(1+x)}dx=\int_0^{1}\frac{1}{\sqrt{x}(1+x)}dx+\int_1^{\infty}\frac{1}{\sqrt{x}(1+x)}dx$

$\int_0^{\infty}\frac{1}{\sqrt{x}(1+x)}dx=\int_0^{1}\frac{1}{\sqrt{x}(1+x)}dx+\int_1^{\infty}\frac{1}{\sqrt{x}(1+x)}dx=\lim_{s\to 0^+}\int_s^{1}\frac{1}{\sqrt{x}(1+x)}dx+\lim_{t\to\infty}\int_1^{t}\frac{1}{\sqrt{x}(1+x)}dx$

令 $u=\sqrt{x}\Rightarrow du=\frac{1}{2\sqrt{x}}dx$，則 $\int\frac{1}{\sqrt{x}(1+x)}dx=2\int\frac{du}{1+u^2}=2\tan^{-1}u+C=2\tan^{-1}\sqrt{x}+C$

$\lim_{s\to 0^+}\int_s^{1}\frac{1}{\sqrt{x}(1+x)}dx+\lim_{t\to\infty}\int_1^{t}\frac{1}{\sqrt{x}(1+x)}dx=2(\tan^{-1}\sqrt{1}-\lim_{s\to 0^+}\tan^{-1}\sqrt{x}+\lim_{t\to\infty}\tan^{-1}\sqrt{x}-\tan^{-1}\sqrt{1})$

$$=2(\frac{\pi}{4}-0+\frac{\pi}{2}-\frac{\pi}{4})=\pi$$

51. (a) Show that $\int_{-\infty}^{\infty} x dx$ is divergent.

(b) Show that $\lim_{t\to\infty}\int_{-t}^{t} x dx = 0$ This shows that we can't define $\int_{-\infty}^{\infty} f(x)dx = \lim_{t\to\infty}\int_{-t}^{t} x dx$

(a) $\int_{-\infty}^{\infty} x dx = \int_{-\infty}^{0} x dx + \int_{0}^{\infty} x dx = -\infty + \infty$ (不定型) 積分發散

(b) $\lim_{t\to\infty}\int_{-t}^{t} x dx = \lim_{t\to\infty}\left[\frac{x^2}{2}\right]_{-t}^{t} = 0$ 與(a)結果不同，故 $\int_{-\infty}^{\infty} f(x)dx \neq \lim_{t\to\infty}\int_{-t}^{t} x dx$

52. If $\int_{-\infty}^{\infty} f(x)dx$ is convergent and a and b are real numbers, show that

$$\int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx = \int_{-\infty}^{b} f(x)dx + \int_{b}^{\infty} f(x)dx$$

設 $a < b$ ，則 $\int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx = \lim_{s\to-\infty}\int_{s}^{a} f(x)dx + \lim_{t\to\infty}\int_{a}^{t} f(x)dx$

$$= \lim_{s\to-\infty}\int_{s}^{a} f(x)dx + \left[\int_{a}^{b} f(x)dx + \lim_{t\to\infty}\int_{b}^{t} f(x)dx\right]$$

$$= \left[\lim_{s\to-\infty}\int_{s}^{a} f(x)dx + \int_{a}^{b} f(x)dx\right] + \lim_{t\to\infty}\int_{b}^{t} f(x)dx$$

$$= \lim_{s\to-\infty}\int_{s}^{b} f(x)dx + \lim_{t\to\infty}\int_{b}^{t} f(x)dx$$

$$= \int_{-\infty}^{b} f(x)dx + \int_{b}^{\infty} f(x)dx$$

廖彥法 整理